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Approximate Factorization for a Class of Second-Order Matrix Functions

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Abstract—The notion of approximate factorization of a Hölder matrix function on a simple smooth closed contour is introduced. The elements of a (2×2)-matrix function of a special form are approximated by polynomials in z and $1/z$. Sufficient conditions for the vanishing of the partial indices of the approximating matrix function are found. Factorization of the approximating matrix function is explicitly constructed.

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Basic notions and definitions. Let Γ be a simple smooth closed contour dividing the plane of a complex variable into two domains D^+ and D^- $(\infty\in D^-)$. By a *factorization* of an H_μ-continuous matrix-function $G(t)$ on Γ we understand the representation of this function in the form $G(t)=G^+(t)G^-(t)$, $t\in\Gamma$, where $G^\pm(t)$ are the limit values in the corresponding domains of a certain matrix function $G(z)$ such that its elements are piecewise analytic and may have a polar singularity at infinity, $\det G(z)\neq 0$ on the finite part of the plane, and at infinity, the order of $\det G^-(z)$ equals the sum of the orders $\varkappa_1,\varkappa_2,\dots,\varkappa_n$ of the rows of the matrix function $G^-(z)$. This definition of factorization differs in form from that given in [1, Part I, p. 59] is that we do not distinguish the middle diagonal factor $\operatorname{diag}\{t^{\varkappa_1},\dots,t^{\varkappa_n}\}$; our notion is closer in meaning to the notion of the *canonical matrix* $X(z)=\{G^+(z), z\in D^+; [G^-(z)]^{-1}, z\in D^-\}$ [2, p. 30]. Taking out the orders of the rows of the matrix function $G^-(z)$ at infinity, we arrive at the conventional definition of factorization, namely,

$$G(t)=G^+(t)\Lambda(t)G^-(t),\quad \Lambda(t)=\operatorname{diag}\{t^{\varkappa_1},\dots,t^{\varkappa_n}\},\tag{1}$$

where $\det G(z)$ does not vanish on the finite part of the plane and takes a nonzero finite value at infinity. The matrix functions $G^+(t)$, $G^-(t)$, and $\Lambda(t)$ in this representation are called the (extreme and middle) *factorization multipliers*. The integers $\varkappa_1\geq\varkappa_2\geq\ldots\geq\varkappa_n$ are referred to as the (left) *partial indices*, and their sum $\varkappa=\operatorname{ind}\det G(t)$, as the *total index* of the matrix function $G(t)$. Representation (1) itself is sometimes called the *left factorization* of the matrix function $G(t)$, as distinct from the representation $G(t)=G^-(t)\Lambda_1(t)G^+(t)$, $\Lambda_1(t)=\operatorname{diag}\{t^{\widetilde\varkappa_1},\dots,t^{\widetilde\varkappa_n}\}$, which is called the *right factorizaton* of $G(t)$ with right partial indices $\widetilde\varkappa_1,\dots,\widetilde\varkappa_n$. At present, the problem of constructing an *approximate factorization* of a matrix function has not been solved in the general case. The main reason for this is that partial indices are unstable with respect to small variations of the matrix function. Under the a priori assumption that the left and right partial indices of the matrix function are zero, the problem of constructing an approximate factorization of this function and the close problem of approximately solving the characteristic system of singular integral equations was considered in [4–7] and other works, and for a Hermitian positive definite matrix function, these problems were considered in [8]. We use the following definition.

Definition. We say that the factorization of an H_μ-continuous matrix function $F(t)$ is an approximate factorization of an H_μ-continuous matrix function $G(t)$ if the matrix functions $F(t)$ and $G(t)$ are close (in the norm of $H_\mu(\Gamma)$) and their partial indices coincide.

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